- There are four hours available for the problems.
- Each problem is worth 10 points.
- Be clear when using a theorem; cite a source if necessary.
- Use a different sheet for each exercise
- Clearly write DRAFT on any draft page you hand in.

MOAWOA

28 April 2017

Problem 1. Given an integer $n \geq 3$, determine the smallest possible value of k such that $\mathbb{R}^k$ contains vectors $u_1, u_2, \ldots, u_n$ satisfying $u_i \cdot u_j = 0 \iff |i-j| > 1$.

Problem 2.

(a) A mathematician wants to tile his garden. He has enough tiles in $k \geq 2$ colours. He wants to put $n \geq 2$ tiles in a row, in such a way that no two tiles that are touching have the same colour. How many ways does this mathematician have to tile his garden?

(b) Another mathematician wants to put tiles around a circular pond in his garden. He needs $n \geq 2$ tiles and has $k \geq 2$ colours too. This time the first and the last tile touch and yet again no two touching tiles are allowed to have the same colour. How many ways does this mathematician have to tile his pond?

Problem 3. Let $(G, \times)$ be a group. For any two group homomorphisms $f, g : G \to G$ we define the maps $f \circ g : x \mapsto f(g(x))$ and $f \cdot g : x \mapsto f(x) \times g(x)$. Also, a homomorphism $f : G \to G$ is called *remarkable* when $f \cdot f = f \circ f$.

(a) Proof the following: if G is abelian, then $f \cdot g$ and $f \circ g$ are homomorphisms for any two homomorfisms $f, g : G \to G$. Is the converse also true?

(b) Determine the the number of natural numbers $n \in \mathbb{Z}_{>0}$ such that $n \leq 50$ and there exist exactly two remarkable homomorphisms $C_n \to C_n$. Here C_n is the cyclic group of order n.

Problem 4. Determine all nonnegative integers n for which there is a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ for which there are n reals r such that the equation $f'(x) = r$ has a real solution x, but the equation $\frac{f(y)-f(z)}{y-z} = r$ has no real solutions $y \neq z$.

Turn Page

Problem 5. Let $(f_n)_{n\geq 0}$ be a sequence of functions $f:\mathbb{C}\to\mathbb{C}$ such that $f_0(z)=0\neq f_1(z)$ for all $z\in\mathbb{C}$ and $2f_{n+2}+2^{-n}zf_{n+1}(z)+f_n(z)=0$ for all $n\geq 0$ and all $z\in\mathbb{C}$. Prove that any root of f_{2017} is real.

Problem 6. Let a_1,a_2,b_1,b_2 be positive numbers such that $\gcd(a_1,a_2)=\gcd(b_1,b_2)=1$ en $a_1\neq\pm a_2$. Let $(x_n)_{n\geq 0}$ be the sequence of numbers given by $x_n=b_2a_1^n-b_1a_2^n$. Suppose $x_n\neq 0$ for all $n\in\mathbb{Z}_{\geq 0}$. Define $S:=\{p \text{ prime} : \exists n \text{ s.t. } p\mid x_n\}$. Prove that S is infinite.